

One-dimensional granular system with memory effects

C. Perrin*, M. Westdickenberg†

March 20, 2017

Abstract

We consider a hybrid compressible/incompressible system with memory effects introduced by Lefebvre Lepot and Maury [14] for the description of one-dimensional granular flows. We prove a first global existence result for this system without additional viscous dissipation. Our approach extends the one by Cavalletti et al. [9] for the pressureless Euler system to the constraint granular case with memory effects. We construct Lagrangian solutions based on an explicit formula of the monotone rearrangement associated to the density and explain how the memory effects are linked to the external constraints imposed on the flow. This result is finally extended to a heterogeneous maximal density constraint depending on time and space.

Keywords : granular flows, pressureless gas dynamics, monotone rearrangement.

MSC : 35Q35, 49J40, 76T25.

Introduction

In this paper, we consider a one-dimensional model for immersed granular flows, introduced by Lefebvre-Lepot and Maury in [14]. The model consists in a system of partial differential equations describing the solid/liquid mixture through the evolution of the density of solid particles ρ and the velocity u

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (1a) \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f & (1b) \\ \partial_t \gamma + \partial_x(\gamma u) = -p & (1c) \\ 0 \leq \rho \leq 1 & (1d) \\ (1 - \rho)\gamma = 0, \quad \gamma \leq 0, & (1e) \end{cases}$$

where p represents the pressure, f an external force. The system also involves the quantity γ , called *adhesion potential*, which is linked to the pressure p through Equation (1c). The system (1a)–(1e) couples the two dynamics that occur in a granular flow: pressureless compressible dynamics in the free zones where $\rho < 1$ and, due to (1e) and (1c), $\gamma = p = 0$; and the

*Institut für Mathematik, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany; perrin@instmath.rwth-aachen.de

†Institut für Mathematik, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany; mwest@instmath.rwth-aachen.de

incompressible dynamics, $\partial_x u = 0$, in the congested parts of the domain where $\rho = 1$ and γ, p are activated. In these congested regions, Equation (1c) models memory effects: the potential γ keeps track of the history of the constraints undergone by the system.

These memory effects have been brought to light by Maury [15] in the case of a single solid particle. More precisely, Maury studies a physical system formed by a vertical wall and a spherical solid particle immersed in a viscous liquid. The particle evolves along the horizontal axis and is submitted to an external force and to the lubrication force exerted by the liquid. The latter becomes predominant when the particle is getting closer to the wall. At first order when the distance q between the particle and the wall goes to 0, it takes the form $F_{\text{lub}} = -C\eta \frac{\dot{q}}{q}$, with η the viscosity of the liquid, $C > 0$ a constant that depends on the diameter of the particle. It prevents the contact in finite time of particle and wall.

Considering the limit of vanishing liquid viscosity $\eta = \varepsilon \rightarrow 0$, Maury proved in [15] the convergence toward a hybrid system (see the system (15) below) describing the two possible states of the system: *free* when $q > 0$, and *stuck* when $q = 0$. At the limit, the system involves a new variable γ , the adhesion potential, which is the residual effect of the singular lubrication force $F_{\text{lub}}^\varepsilon$ at the limit. The potential characterizes stickyness of the particle: even in case of a pulling external force, it may take some time before the particle takes off from the wall.

Lefebvre-Lepot and Maury have extended this idea to a one-dimensional macroscopic system of aligned solid particles: the system (1) is introduced in [14] as the formal limit of the system

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 & (2a) \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho u_\varepsilon^2) - \partial_x\left(\frac{\varepsilon}{1-\rho_\varepsilon} \partial_x u_\varepsilon\right) = \rho_\varepsilon f & (2b) \end{cases}$$

The lubrication force is represented at this macroscopic scale by the singular viscous term $\partial_x\left(\frac{\varepsilon}{1-\rho_\varepsilon} \partial_x u_\varepsilon\right)$ which prevents, by analogy with the single particle case, the formation of congested domains $\rho = 1$ when $\varepsilon > 0$. However, the rigorous proof of the convergence of the solutions of (2) to solutions of (1) remains an open problem. The mathematical difficulty of this singular limit relies in the lack of compactness of the non-linear term $\rho_\varepsilon u_\varepsilon^2$. This kind of singular limit has nevertheless been proved in [18] (see also [19] for a result in dimension 2) on an augmented system where an additional physical dissipation is taken into account.

We intend in this paper to prove the first global existence result of weak solutions for the system (1). We want to build Lagrangian solutions to the system (1) directly, without any lubrication approximation. For that purpose, we take advantage of the link between model (1) and the model of pressureless gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (3a) \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 & (3b) \end{cases}$$

Indeed, by a formal differentiation of Equation (1c) and use of the incompressibility constraint $\partial_x u = 0$ in the congested domain we get

$$\partial_t \partial_x \gamma + \partial_x(u \partial_x \gamma) = -\partial_x p$$

which can be also rewritten with the exclusion relation (1e) as

$$\partial_t(\rho \partial_x \gamma) + \partial_x(\rho u \partial_x \gamma) = -\partial_x p. \quad (4)$$

The subtraction of (1b) by this equation cancels then the pressure p and the resulting system is a pressureless system with external force f and two velocities $u, v = u - \partial_x \gamma$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (5a) \\ \partial_t(\rho v) + \partial_x(\rho uv) = \rho f & (5b) \\ v = u - \partial_x \gamma & (5c) \\ 0 \leq \rho \leq 1 & (5d) \\ (1 - \rho)\gamma = 0, \quad \gamma \leq 0 & (5e) \end{cases}$$

Among the large literature that exists for the pressureless system (3), we are interested in the recent results of Natile, Savaré [17] and Cavalletti et al. [9] that develop a Lagrangian approach based on the representation of the density ρ by its monotone rearrangement X which is the optimal transport between the Lebesgue measure $\mathcal{L}_{[0,1]}^1$ and ρ (see [22]).

Let us finally mention that the granular system (1) can also be seen as a non-trivial extension of the pressureless Euler equations under maximal density constraint

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (6a) \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \pi = 0 & (6b) \\ 0 \leq \rho \leq 1 & (6c) \\ (1 - \rho)\pi = 0, \quad \pi \geq 0 & (6d) \end{cases}$$

This system has been first introduced by Bouchut et al. in [5] for the modelling of two-phase flows and then studied by Berthelin [3] and Wolansky [24], both studies relying on a discrete approximation generalizing the sticky particle dynamics used for the pressureless system. Recently, numerical methods based on optimal transport tools have also been developed on this system (see [16] and [21]).

For viscous fluids, i.e., Navier-Stokes systems, a theoretical existence result can be found in [20] in the case where the maximal density constraint ρ^* is a given function of the space variable $\rho^*(x)$. Recently, Degond et al. have proved in [10] the existence of global weak solutions to the Navier-Stokes system with a time and space dependent maximal constraint $\rho^*(t, x)$ which is transported by the velocity u

$$\partial_t \rho^* + u \partial_x \rho^* = 0. \quad (7)$$

This type of heterogeneous maximal constraint may be relevant in particular for the dynamics of floating structures (see for instance Lannes [12]).

The paper is organized as follows: in Section 1 we briefly review the literature on the pressureless gas dynamics and introduce the mathematical tools linked to a Lagrangian description. In Section 2 we explain formally how these tools can be extended to the study of system (1) and give our main existence result. Section 3 is devoted to the proof of this result and Section 4 presents some numerical simulations. We extend finally the result in the last section to the special case of time and space dependent maximal density constraint satisfying (7).

1 Lagrangian approach for the pressureless Euler equations

The pressureless gas dynamics equations, augmented by the assumption of adhesion dynamics, has been proposed as a simple model for the formation of large scale structures in the universe such as aggregates of galaxies. It is linked to the sticky particle system introduced by Zeldovich in [25]. Since the work of Bouchut [4], which points out the obstacles to proving existence of classical solutions to (3) (concentration phenomena on the density, lack of uniqueness under classical entropy conditions), several different mathematical approaches have been proposed in the literature to prove the global existence of measure solutions under suitable entropy conditions (see [4]), among which: approximations by the discrete sticky particles dynamics (see for instance [8] and [17]), approximation by viscous regularization (see [23] and [6]) or, more recently, derivation by a hydrodynamic limit (see [11]).

In particular, Natile and Savaré use in [17] a very interesting Lagrangian characterization of the density ρ by its monotone rearrangement X to show the convergence of the discrete sticky particle system as the number N of particles goes to $+\infty$. To every probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ (i.e., with finite quadratic moment $\int_{\mathbb{R}} |x|^2 d\rho(x) < +\infty$) is associated a unique transport $X \in K$, the closed convex cone of non-decreasing maps in $L^2(0, 1)$, such that

$$\rho_t = (X_t)_\# \mathcal{L}_{[0,1]}^1. \quad (8)$$

Here $\mathcal{L}_{[0,1]}^1$ is the one-dimensional Lebesgue measure on $[0, 1]$ and $\#$ denotes the *push-forward* of measures, defined for all Borel maps $\zeta : \mathbb{R} \rightarrow [0, \infty]$ by

$$\int_{\mathbb{R}} \zeta(x) d\rho_t(x) = \int_0^1 \zeta(X_t(y)) dy. \quad (9)$$

If (ρ, u) is a solution in the distributional sense of (3), then u_t can be associated to the Lagrangian velocity $U_t := \dot{X}_t$ (in the sequel all the Lagrangian variables will be denoted by capital letters and the Eulerians ones by the corresponding small letters) through

$$U_t(y) = u_t(X_t(y)). \quad (10)$$

In [17], Natile and Savaré show different characterizations of the transport X associated to an Eulerian solution of (3), in particular they prove that

$$X_t = P_K(X_0 + tU_0) \quad \text{for all } t \geq 0. \quad (11)$$

The map $X_0 + tU_0$ represents the free motion path, which is at the discrete level the transport corresponding to the case where the particles do not interact at all.

These arguments have been then extended by Brenier et al. in [7] to systems including an interaction between the discrete particles. This interaction is represented at the continuous level by a force $f(\rho)$ in the right-hand side of the momentum equation (3b).

Recently, Cavalletti et al. [9] have taken advantage of the formula (11) to construct directly global weak solutions to (3) without any discrete approximation by sticky particles. To this end, they define for all positive times t the transport X_t associated to an initial data (ρ_0, u_0) by equation (11). The Lagrangian variables X_0 and U_0 are then defined as

$$\rho_0 = (X_0)_\# \mu, \quad U_0 := u_0 \circ X_0 \quad (12)$$

for a more general reference measure μ in $\mathcal{P}_2(\mathbb{R})$ (for instance $\mu = \rho_0$ and in this case $X_0 = \text{id}$). Due the contraction property of the projection operator P_K , one ensures that the map $t \mapsto X_t$ is Lipschitz continuous and thus differentiable for a.e. t , which allows us to define the Lagrangian velocity $U_t := \dot{X}_t$. Cavalletti et al. introduce next the subspace in $\mathcal{L}^2(\mathbb{R}, \mu)$ formed by functions which are essentially constant where X_t is constant:

$$\mathcal{H}_{X_t} = \mathcal{L}^2(\mathbb{R}, \mu) \text{-closure of } \{\varphi \circ X_t, \varphi \in \mathcal{D}(\mathbb{R})\}, \quad (13)$$

This space is a subset of the tangent cone to K at X_t , $\mathbb{T}_{X_t}K$, in which the Lagrangian velocity is contained. The authors prove that U_t is the orthogonal projection of U_0 onto \mathcal{H}_{X_t} :

$$U_t = P_{\mathcal{H}_{X_t}}(U_0). \quad (14)$$

This property ensures that there exists for a.e. t an Eulerian velocity $u_t \in \mathcal{L}^2(\mathbb{R}, \rho_t)$ such that $U_t = u_t \circ X_t$. This is the key argument to recover the weak formulations of the Eulerian equations (3a)–(3b).

By comparison, our granular system written under the pressureless form (5) involves an additional maximal density constraint $\rho \leq 1$, an additional variable γ linked to this maximal constraint and leading to consider the modified velocity v , and an external force f .

We explain in the next section how to extend the previous tools in order to deal with these additional constraints and variables.

2 Extension to granular flows, main result

Before announcing our main existence theorem, it is necessary to explain how to adapt the previous Lagrangian tools in presence of an external force and a maximal density constrained. A good way to do this is to come back to the microscopic approach, by nature Lagrangian, developed by Maury in [15] for a single sticky particle in contact with a wall.

Single particle case. In his study [15], Maury proves by a vanishing viscosity limit (viscosity of liquid in which the particle is immersed), the existence of solutions to the hybrid system

$$\begin{cases} \dot{q} + \gamma = u_0 + \int_0^t f(s)ds \\ q \geq 0, \quad \gamma \leq 0, \quad q\gamma = 0, \end{cases} \quad (15a)$$

$$(15b)$$

which describes the two possibles states of the system: *free* when $q > 0$ (the particle evolves freely under the external force f), and *stuck* when $q = 0$. In this latter case, the adhesion potential is activated and is equal to

$$u^{\text{free}} = u_0 + \int_0^t f(s)ds, \quad (16)$$

which is the velocity that the particle would have if there was no wall on its trajectory. The system (15) is in fact equivalent to the following second order system (see [13])

$$\begin{cases} m\ddot{q} = mf + \lambda & (17a) \\ \dot{q}(t^+) = P_{C_{q,\gamma}(t)}\dot{q}(t^-) & (17b) \\ \text{spt}(\lambda) \subset \{t: q(t) = 0\} & (17c) \\ \dot{\gamma} = -\lambda & (17d) \\ q \geq 0, \quad \gamma \leq 0 & (17e) \end{cases}$$

where m is the mass of the particle and $C_{q,\gamma}(t)$ denotes the set of admissible velocities

$$C_{q,\gamma}(t) = \begin{cases} \{0\} & \text{if } \gamma(t^-) < 0 \\ \mathbb{R}^+ & \text{if } \gamma(t^-) = 0, q(t) = 0 \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

It ensures that the particle cannot cross the wall and that it sticks to the wall as long as $\gamma < 0$. By comparison with the macroscopic system (1), an analogy can be made between the variables q and $1 - \rho$, between λ and $-\partial_x p$ and thus between γ_{micro} defined by (17d) and $-\partial_x \gamma_{\text{macro}}$.

Extension of the Lagrangian approach. Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ be the initial density. We assume that ρ_0 is absolutely continuous with respect to the Lebesgue measure and that its density (also denoted ρ_0) satisfies the maximal constraint

$$0 \leq \rho_0 \leq 1 \quad \text{a.e.} \quad (18)$$

As suggested by Cavalletti et al. [9] we set $\mu = \rho_0$ as reference measure. In this paper, the set of square-integrable functions with respect to this reference measure will be denoted $\mathcal{L}^2(\mathbb{R}, \rho_0)$. Endowed with its natural scalar product denoted by $\langle \cdot, \cdot \rangle$, it is a Hilbert space. By comparison, the space of p -integrable functions on the domain Ω for the Lebesgue measure will be denoted by $L^p(\Omega)$.

Set of admissible transports. In order to express the maximal density constraint in terms of a constraint on the transport X_t associated to ρ_t , such that

$$\rho_t = (X_t)_\# \rho_0, \quad (19)$$

we define the maximally compressed density starting from the initial state ρ_0 which is a characteristic function of an interval of length one. To fix it, we assume that this interval is centred in the mean value of ρ_0 , but as we shall see it can be easily translated. Let $\tilde{\rho}$ be the probability measure associated to this characteristic function and define \tilde{X} as the unique nondecreasing transport map of $\mathcal{L}^2(\mathbb{R}, \rho_0)$ (see for instance [22] Theorem 2.5) such that

$$\tilde{\rho} = \tilde{X}_\# \rho_0. \quad (20)$$

The push forward formula (see [1] Lemma 5.5.3) ensures on the one hand that $\partial_y \tilde{X} > 0$ a.e., and on the other hand that ρ_t is absolutely continuous with respect to the Lebesgue measure if and only if the approximate derivative $\partial_y X_t$ is positive a.e. In this case for a.e. $x \in \mathbb{R}$

$$\rho_t(x) = \frac{\rho_0(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))} = \frac{\rho_0(X_t^{-1}(x))}{\partial_y \tilde{X}_t(X_t^{-1}(x))} \frac{\partial_y \tilde{X}_t(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))} = \frac{\partial_y \tilde{X}_t(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))} \quad (21)$$

To guarantee the maximal density constraint we are thus led to consider the transports X_t such that $\partial_y \tilde{X}_t \leq \partial_y X_t$ a.e. We introduce then the closed convex set of admissible transports in $\mathcal{L}^2(\mathbb{R}, \rho_0)$ that ensure the maximal density constraint. We define

$$\tilde{K} := K + \tilde{X} \quad (22)$$

where we recall that K is the cone of non-decreasing maps of $\mathcal{L}^2(\mathbb{R}, \rho_0)$. To the transport $X_t \in \tilde{K}$ we associate the monotone transport

$$S_t := X_t - \tilde{X}. \quad (23)$$

We recall that for any $S \in K$ the tangent cone to K at S is defined as

$$\mathbb{T}_S K := \mathcal{L}^2(\mathbb{R}, \mu) \text{-closure of } T_S K \quad \text{where} \quad T_S K := \bigcup_{h>0} h(K - S). \quad (24)$$

Remark: Coming back to the definition of $\tilde{\rho}$, we observe that the position of the interval does not matter for the definition of \tilde{K} since the translations can be absorbed in K .

Definition of the transport X_t . To define for all time t an appropriate transport X_t , we need to extend the notion of free transport $X_0 + tU_0$ used in (11) to the case where the external force f is applied on the system. In particular we need to extend the notion of free velocity, which for the pressureless system is simply the initial velocity U_0 . In our case, inspired by the microscopic case (16), we set

$$U_t^{\text{free}} := U_0 - \partial_y \Gamma_0 + \int_0^t f(s, X_s) ds \quad (25)$$

where we take also into account the initial Lagrangian adhesion potential if the initial density is congested in some part of the domain. The free trajectory at time t is then defined as

$$X_t^{\text{free}} := X_0 + \int_0^t U_s^{\text{free}} ds = \text{id} + \int_0^t U_s^{\text{free}} ds. \quad (26)$$

We then define (see [7] for a similar formulation)

$$X_t := P_{\tilde{K}}(X_t^{\text{free}}) = P_{\tilde{K}}\left(\text{id} + \int_0^t U_s^{\text{free}} ds\right). \quad (27)$$

By analogy with the microscopic case (15a), we also want to set

$$\partial_y \Gamma_t := U_t - U_t^{\text{free}} \quad \text{where} \quad U_t = \dot{X}_t. \quad (28)$$

Definition of weak solutions and main result. We observe that it is $V_0 := U_0 - \partial_y \Gamma_0$ that is involved in (25) and, as explained before, the introduction of the velocity $V_t = U_t - \partial_y \Gamma_t$ allows us to link our analysis with that of the classical pressureless dynamics. It is then natural to work on the formulation (5) rather than (1).

Definition 1 *A triplet (ρ, u, γ) is called a weak solution of system (5) if*

- *the initial data is attained weakly as $t \rightarrow 0$:*

$$\rho_t \rightharpoonup \rho_0, \quad v_t = u_t - \partial_x \gamma_t \rightharpoonup u_0 - \partial_x \gamma_0;$$

- *Equations (5a) and (5b) are satisfied in the sense of distributions;*
- *Equations (5d) and (5e) hold almost everywhere;*
- *velocity and adhesion potential satisfy*

$$u_t \in \mathcal{L}^2(\mathbb{R}, \rho_t), \quad \gamma_t \in W^{1,2}(\mathbb{R}) \quad \text{for a.a. } t. \quad (29)$$

Theorem 1 *Let $T > 0$, $f \in L^\infty(0, T; \text{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ and $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ with $\rho_0 \ll \mathcal{L}^1$ and $0 \leq \rho_0 \leq 1$ a.e. Assume that $(u_0, \gamma_0) \in \mathcal{L}^2(\mathbb{R}, \rho_0) \times W^{1,2}(\mathbb{R})$ with $(1 - \rho_0)\gamma_0 = 0$ a.e. Let*

$$X_0 := \text{id}, \quad U_0 := u_0 \circ X_0, \quad \Gamma_0 := \gamma_0 \circ X_0$$

and define for all $t \in [0, T]$ the coupled variables U_t^{free} and X_t by (25)–(27). Then $t \mapsto X_t$ is differentiable for a.e. $t \in (0, T)$ and we can define

$$U_t(y) := \dot{X}_t(y), \quad \Gamma_t(y) := \int_{-\infty}^y \left(U_t(z) - U_t^{\text{free}}(z) \right) d\rho_0(z).$$

There exist $(u_t, \gamma_t) \in \mathcal{L}^2(\mathbb{R}, \rho_t) \times W^{1,2}(\mathbb{R})$, where $\rho_t := (X_t)_\# \rho_0$, such that

$$U_t = u_t \circ X_t, \quad \Gamma_t = \gamma_t \circ X_t.$$

The triplet (ρ, u, γ) is a global weak solution of system (5).

Remark: The assumption $f \in L^\infty(0, T; \text{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ can certainly be relaxed to include a larger class of forces. We will stick to it here to simplify the presentation.

3 Construction of global weak solutions

Let us begin by justifying the fact that we can define in a unique manner X_t for all times.

Lemma 1 *For all $t \in [0, T]$, there exists a unique solution (U_t^{free}, X_t) to (25)–(27).*

Proof. Let $E = \mathcal{C}([0, T], \mathcal{L}^2(\mathbb{R}, \rho_0))$ endowed with the norm

$$\|X\|_{\mathcal{E}} = \max_{t \in [0, T]} e^{-2\sqrt{k}t} \|X_t\|_{\mathcal{L}^2_{\rho_0}}$$

where k is the Lipschitz constant associated to the external force f . We define the application \mathcal{T} by

$$\mathcal{T}(X)(t) = P_{\tilde{K}} \left(X_0 + t(U_0 - \partial_y \Gamma_0) + \int_0^t \int_0^\tau f(s, X_s) ds d\tau \right), \quad t \in [0, T].$$

To prove the existence of a unique solution to (25)–(27) we need to show that the application \mathcal{T} is a contraction on \mathcal{E} . Consider $X^1, X^2 \in \mathcal{E}$ starting at $t = 0$ from X_0 with velocity $U_0 - \partial_y \Gamma_0$. Thanks to the contraction property of the projection map we have

$$\begin{aligned} \|\mathcal{T}(X^1)(t) - \mathcal{T}(X^2)(t)\|_{\mathcal{L}^2_{\rho_0}} &\leq \left\| \int_0^t \int_0^\tau (f(s, X_s^1) - f(s, X_s^2)) ds d\tau \right\|_{\mathcal{L}^2_{\rho_0}} \\ &\leq \int_0^t \int_0^\tau \|f(s, X_s^1) - f(s, X_s^2)\|_{\mathcal{L}^2_{\rho_0}} ds d\tau \\ &\leq k \int_0^t \int_0^\tau \|X_s^1 - X_s^2\|_{\mathcal{L}^2_{\rho_0}} ds d\tau \\ &\leq k \|X^1 - X^2\|_{\mathcal{E}} \int_0^t \int_0^\tau e^{2\sqrt{k}s} ds d\tau \\ &\leq \frac{1}{4} e^{2\sqrt{k}t} \|X^1 - X^2\|_{\mathcal{E}}. \end{aligned}$$

We have therefore

$$\|\mathcal{T}(X^1) - \mathcal{T}(X^2)\|_{\mathcal{E}} \leq \frac{1}{4} \|X^1 - X^2\|_{\mathcal{E}},$$

which proves by the Banach Fixed Point Theorem that there exists a unique application X_t solution to (27) and thus a unique U_t^{free} for all times. \square

We recall now two useful lemmas proved in [9].

Lemma 2 (Lemma 3.1 [9]) *For given $S \in \mathcal{L}^2(\mathbb{R}, \rho_0)$ monotone, we define $\Pi_S := (\text{id}, S)_{\#} \rho_0$. Then there exists a Borel set N_S such that $\rho_0(N_S) = 0$ and*

$$(y, S(y)) \in \text{spt } \Pi_S \quad \text{for all } y \in \mathbb{R} \setminus N_S.$$

Lemma 3 (Lemma 3.6 [9]) *Let N_{S_t} be the ρ_0 -null set associated to S_t by the previous lemma. We define*

$$L_t^x := \{y \in \mathbb{R} \setminus N_{S_t}, S_t(y) = x\}$$

and

$$\mathcal{O}_t = \{x \in \mathbb{R}, L_t^x \text{ has more than one element}\}.$$

The set \mathcal{O}_t is at most countable and S_t is injective on $\mathbb{R} \setminus \bigcup_{x \in \mathcal{O}_t} L_t^x$.

The domain \mathcal{O}_t is therefore the congested domain at time t . For simplicity, we set

$$\Omega_{S_t} := \bigcup_{x \in \mathcal{O}_t} L_t^x. \quad (30)$$

An admissible velocity U has to be constant on this congested domain, this is why we introduce the following space of velocities

$$\mathcal{H}_{S_t} := \{U \in \mathcal{L}^2(\mathbb{R}, \rho_0) \mid U \text{ is a.e. constant on maximal intervals in } \Omega_{S_t}\}. \quad (31)$$

Proposition 1 *The velocity $U_t = \frac{d}{dt}X_t$ exists for a.e. $t \in (0, T)$ and for such t belongs to the space \mathcal{H}_{S_t} .*

Proof. Due to the contraction property of the projection, we have

$$\begin{aligned} \|X_{t+h} - X_t\|_{\mathcal{L}_{\rho_0}^2} &\leq \left\| \int_0^t U_s^{\text{free}} ds \right\|_{\mathcal{L}_{\rho_0}^2} \\ &\leq h \|V_0\|_{\mathcal{L}_{\rho_0}^2} + \left\| \int_t^{t+h} \left(\int_0^s f(\tau, X_\tau) d\tau \right) ds \right\|_{\mathcal{L}_{\rho_0}^2} \end{aligned}$$

and since $f \in L^\infty(0, T; L^\infty(\mathbb{R}))$ we deduce that

$$\|X_{t+h} - X_t\|_{\mathcal{L}_{\rho_0}^2} \leq |h| \left(\|V_0\|_{\mathcal{L}_{\rho_0}^2} + C(\|f\|_{L^\infty}) \right). \quad (32)$$

This proves that $t \mapsto X_t$ is Lipschitz continuous and its time-derivative exists strongly for a.e. $t \in (0, T)$

$$U_t = \lim_{h \rightarrow 0^+} \frac{X_{t+h} - X_t}{h} = - \lim_{h \rightarrow 0^+} \frac{X_{t-h} - X_t}{h}.$$

We deduce that

$$U_t \in \mathbb{T}_{S_t} K \cap (-\mathbb{T}_{S_t} K).$$

Now $U_t \in \mathbb{T}_{S_t} K$ implies that there exist two sequences (W_t^k) , (λ^k) with $W_t^k \in K$ and $\lambda^k > 0$, such that

$$U_t^k = W_t^k - \lambda^k S_t \quad \text{converges strongly to } U_t \text{ in } L^2(\mathbb{R}).$$

We can then extract a subsequence, still denoted (U_t^k) , which converges a.e. towards U_t . For every k we denote by N^k the ρ^0 -null set associated to W_t^k ; see Lemma 2. There exists a $B \subset \mathbb{R}$ with $\rho_0(B) = 0$, such that $\bigcup_k N^k \subset B$ and

$$U_t^k(y) \xrightarrow[k \rightarrow +\infty]{} U_t(y) \quad \forall y \in \mathbb{R} \setminus B.$$

For all $x \in \mathcal{O}_t$, $y_1, y_2 \in L_t^x \setminus B$, we have

$$\begin{aligned} (y_1 - y_2)(U_t^k(y_1) - U_t^k(y_2)) &= (y_1 - y_2)(W_t^k(y_1) - \lambda^k S_t(y_1) - W_t^k(y_2) + \lambda^k S_t(y_2)) \\ &= (y_1 - y_2)(W_t^k(y_1) - W_t^k(y_2)), \\ &\geq 0 \end{aligned}$$

by monotonicity of W_t^k , and thus by passing to the limit $k \rightarrow +\infty$

$$(y_1 - y_2)(U_t(y_1) - U_t(y_2)) \geq 0.$$

Using now the fact that $U_t \in (-\mathbb{T}_{S_t}K)$, we obtain in the same way

$$(y_1 - y_2)(U_t(y_1) - U_t(y_2)) \leq 0.$$

Thus

$$(y_1 - y_2)(U_t(y_1) - U_t(y_2)) = 0 \quad \text{for all } x \in \mathcal{O}, \ y_1, y_2 \in L_t^x \setminus B, \quad (33)$$

which means that U_t belongs to \mathcal{H}_{S_t} . \square

Proposition 2 *There exists a velocity $u_t \in \mathcal{L}^2(\mathbb{R}, \rho_t)$ such that*

$$U_t(y) = u_t(X_t(y)) \quad \text{where} \quad \rho_t := (X_t)_\# \rho_0. \quad (34)$$

Proof. Since X_t belongs to \tilde{K} , for all $x \in \mathbb{R}$ there exists at most one $y \in \mathbb{R} \setminus N_{X_t}$ (where N_{X_t} is the null set associated to X_t) with $X_t(y) = x$ we can then set

$$u_t(x) := \begin{cases} U_t(y) & \text{if such } y \text{ exists} \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

We have then

$$U_t(y) = u_t(X_t(y)) \quad \text{for a.e. } y \in \mathbb{R} \setminus N_{X_t}$$

and

$$\|u_t\|_{\mathcal{L}_{\rho_t}^2} = \|U_t\|_{\mathcal{L}_{\rho_0}^2}. \quad \square$$

Lemma 4 *The space \mathcal{H}_{S_t} is characterized as*

$$\mathcal{H}_{S_t} = \{W \in L^2(\mathbb{R}, \rho_0) : \text{there exists } w \in \mathcal{L}^2(\mathbb{R}, \eta_t) \text{ with } W = w \circ S_t\}$$

where $\eta_t := (S_t)_\# \rho_0$.

Proof. Let $W \in \mathcal{H}_{S_t}$. By definition, W is essentially constant on each maximal interval of Ω_{S_t} . For all $x \in \mathbb{R} \setminus \mathcal{O}_t$ there exists at most one $y \in L_t^x \setminus N_{S_t}$ such that $S_t(y) = x$, so we define

$$w(x) := \begin{cases} W(y) & \text{if such } y \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

For all $x \in \mathcal{O}_t$, since W is a.e. constant on L_t^x , we can pick a generic $y \in L_t^x$ and define

$$w(x) := W(y).$$

By doing so, we have constructed w such that

$$W(y) = w(S_t(y)) \quad \text{for -a.e. } x \in \mathbb{R} \setminus N_{S_t}.$$

We have then

$$\begin{aligned} \int_{\mathbb{R}} |W(y)|^2 dy &= \int_{\mathbb{R}} |w(S_t(y))|^2 dy \\ &= \int_{\mathbb{R}} |w(x)|^2 d\eta_t(x). \quad \square \end{aligned}$$

Proposition 3 *The space \mathcal{H}_{S_t} is included in the $\mathcal{L}^2(\mathbb{R}, \rho_0)$ -closure of*

$$T_{S_t}K \cap [X_t^{\text{free}} - X_t]^\perp.$$

Proof. Due to the previous lemma, we are led to show that

$$\varphi \circ S_t \in T_{S_t}K \cap [X_t^{\text{free}} - X_t]^\perp \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}),$$

to get the desired result. We consider $h > \|\varphi\|_{\mathcal{L}_{\rho_0}^\infty}$ and set

$$Z_h^\pm = \left(\text{id} \pm \frac{1}{h}\varphi\right) \circ S_t \in K.$$

We then have

$$\varphi \circ S_t = h(Z_h^+ - S_t),$$

which is by definition an element of $T_{S_t}K$. On the other hand, using the fact that X_t is the projection of X_t^{free} , we get

$$\begin{aligned} \pm \langle X_t^{\text{free}} - X_t, \varphi \circ S_t \rangle &= h \langle X_t^{\text{free}} - X_t, Z_h^\pm - S_t \rangle \\ &= h \langle X_t^{\text{free}} - X_t, \tilde{Z}_h^\pm - X_t \rangle \leq 0 \end{aligned}$$

where $\tilde{Z}_h^\pm = Z_h^\pm + \tilde{X} \in \tilde{K}$, which proves that $\varphi \circ S_t \in [X_t^{\text{free}} - X_t]^\perp$. \square

Proposition 4 *The Lagrangian velocity U_t is the orthogonal projection of U_t^{free} onto \mathcal{H}_{S_t} .*

Proof. We already know that $U_t \in \mathcal{H}_{S_t}$. Let us therefore show that

$$\langle U_t^{\text{free}} - U_t, U_t \rangle = 0 \quad \text{and} \quad \langle U_t^{\text{free}} - U_t, W \rangle \leq 0 \quad \text{for all } W \in \mathcal{H}_{S_t}. \quad (36)$$

- $\langle U_t^{\text{free}} - U_t, U_t \rangle \geq 0$: For any $t, h \in \mathbb{R}$, the quantity

$$\frac{X_{t+h} - X_t}{h}$$

is uniformly bounded. Therefore there exists a sequence (h_n) such that

$$U_t^n := \frac{X_{t+h_n} - X_t}{h_n} \rightharpoonup U_t \quad \text{weakly in } \mathcal{L}^2(\mathbb{R}, \rho_0).$$

Using the fact that X_{t+h_n} is the projection of $X_{t+h_n}^{\text{free}}$ onto \tilde{K} , we have the inequality

$$\langle X_{t+h_n}^{\text{free}} - X_{t+h_n}, X_t - X_{t+h_n} \rangle \leq 0,$$

which can also be rewritten as

$$\langle X_t^{\text{free}} - X_t - h_n U_t^n + \int_t^{t+h_n} U_s^{\text{free}} ds, -h_n U_t^n \rangle \leq 0$$

or, by splitting the powers of h_n , as

$$-h_n \langle X_t^{\text{free}} - X_t, U_t^n \rangle - h_n^2 \langle U_t^n - \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds, U_t^n \rangle \leq 0.$$

Since $h_n U_t^n = X_{t+h_n} - X_t$ and X_t is the projection on \tilde{K} of X_t^{free} , we deduce that the first term of the left-hand side has a positive sign and thus

$$-h_n^2 \langle U_t^n - \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds, U_t^n \rangle \leq 0.$$

As $h_n \rightarrow 0^+$ we have then

$$\frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds \rightarrow U_t^{\text{free}} \quad \text{strongly in } \mathcal{L}^2(\mathbb{R}, \rho_0).$$

From the weak convergence of U_t^n towards U_t , it follows that

$$\langle \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds, U_t^n \rangle \rightarrow \langle U_t^{\text{free}}, U_t \rangle$$

$$\|U_t\|_{L^2}^2 \leq \liminf \|U_t^n\|_{L^2}^2.$$

So we finally obtain the desired inequality

$$\langle U_t^{\text{free}} - U_t, U_t \rangle \geq 0.$$

- $\langle U_t^{\text{free}} - U_t, W \rangle \leq 0$ for all $W \in \mathcal{H}_{S_t}$: Thanks to Propositions 1 and 3, there exists $h > 0$ and $Z_t \in K$ such that

$$W = h(Z_t - S_t) \quad \text{and} \quad \langle X_t^{\text{free}} - X_t, Z_t - S_t \rangle = 0.$$

To prove the inequality, we are led to show that

$$\langle U_t^{\text{free}} - U_t, Z_t - S_t \rangle \leq 0.$$

We consider as before the approximate velocity U_t^n and introduce $\delta_n := U_t^n - U_t$. Since X_{t+h_n} is the projection of $X_{t+h_n}^{\text{free}}$ onto \tilde{K} , we have

$$\begin{aligned} 0 &\geq \langle X_{t+h_n}^{\text{free}} - X_{t+h_n}, Z_t + \tilde{X} - X_{t+h_n} \rangle \\ &= \langle X_t^{\text{free}} - X_t + h_n \left(\frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds - U_t \right) - h_n \delta_n, (Z_t - S_t) - h_n U_t - h_n \delta_n \rangle. \end{aligned}$$

Rearranging the terms, we can then get

$$\begin{aligned} &h_n \langle \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds - U_t, Z_t - S_t \rangle \\ &\leq -\langle X_t^{\text{free}} - X_t, Z_t - S_t \rangle + h_n \langle X_t^{\text{free}} - X_t, U_t \rangle \\ &\quad + h_n \left(\langle X_t^{\text{free}} - X_t, \delta_n \rangle + \langle \delta_n, Z_t - S_t \rangle \right) \\ &\quad + h_n^2 \left(\langle \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds - U_t, U_t \rangle + \langle \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}} ds - U_t, \delta_n \rangle \right) \\ &\quad + h_n^2 \left(\langle \delta_n, U_t \rangle - \|\delta_n\|_{\mathcal{L}^2}^2 \right). \end{aligned}$$

By definition of Z_t and Proposition 3, the first line of the right-hand side vanishes. Dividing now by h_n and letting $h_n \rightarrow 0$, the other terms of the right-hand side tend to 0 and we get

$$\langle U_t^{\text{free}} - U_t, Z_t - S_t \rangle \leq 0. \quad (37)$$

With these two points we conclude that U_t is the orthogonal projection of the free velocity U_t^{free} onto the space \mathcal{H}_{S_t} . \square

Remark: Since U_t is the orthogonal projection of V_0 onto \mathcal{H}_{S_t} , we have

$$U_t(y) = \begin{cases} U_t^{\text{free}}(y) & \text{if } y \in \mathbb{R} \setminus \bigcup_{x \in \mathcal{O}_t} L_t^x \\ \frac{1}{|I|} \int_I U_t^{\text{free}}(z) d\rho_0(z) & \text{if } y \in I, I \in \mathcal{J}(L_t^x) \text{ for some } x \in \mathcal{O}_t \end{cases} \quad (38)$$

where $\mathcal{J}(L_t^x)$ denotes the set of maximal intervals contained in L_t^x .

Recovering of the mass equation. As explained in Section 2 (see Lemma 5.5.3 in [1]), the probability measure ρ_t , defined as the push-forward

$$\rho_t := (X_t)_\# \rho_0, \quad X_t \in \tilde{K}, \quad (39)$$

is absolutely continuous with respect to the Lebesgue measure and satisfies the maximal density constraint

$$0 \leq \rho_t \leq 1 \quad \text{a.e.} \quad (40)$$

Moreover, for all $\xi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R})$ we have by a change a variable

$$\begin{aligned}
& - \int_{\mathbb{R}} \xi(0, y) d\rho_0(y) \\
&= \int_0^T \frac{d}{dt} \left(\int_{\mathbb{R}} \xi(t, X_t(y)) d\rho_0(y) \right) dt \\
&= \int_0^T \int_{\mathbb{R}} \left(\partial_t \xi(t, X_t(y)) + \frac{d}{dt} X_t(y) \partial_x \xi(t, X_t(y)) \right) d\rho_0(y) dt \\
&= \int_0^T \int_{\mathbb{R}} \left(\partial_t \xi(t, X_t(y)) + U_t(y) \partial_x \xi(t, X_t(y)) \right) d\rho_0(y) dt,
\end{aligned}$$

which gives the weak formulation of the mass equation (5c) (see Proposition 2):

$$- \int_{\mathbb{R}} \xi(0, x) d\rho_0(x) = \int_0^T \int_{\mathbb{R}} (\partial_t \xi(t, x) + u_t(x) \partial_x \xi(t, x)) d\rho_t(x) dt. \quad (41)$$

3.1 Memory effects, definition of the adhesion potential

By analogy with the discrete model (15), we define the adhesion potential Γ_t for a.e. $t \in (0, T)$ as

$$\Gamma_t(y) := \int_{-\infty}^y (U_t(z) - U^{\text{free}}(z)) d\rho_0(z). \quad (42)$$

Proposition 5 *For a.e. $t \in (0, T)$, $y \in \mathbb{R}$, we have $\Gamma_t(y) \leq 0$ and $\text{spt } \Gamma_t \subset \bigcup_{x \in \mathcal{O}_t} L_t^x$.*

Proof. For simplicity, we number in increasing order the maximal intervals $I_k \in \mathcal{J}(\bigcup_{x \in \mathcal{O}_t} L_t^x)$. If $y \in \mathbb{R} \setminus \bigcup_{x \in \mathcal{O}_t} L_t^x =: \Omega_t^f$, we set

$$k_y = \sup \left\{ k \in \mathbb{N} \mid y > \sup I_k, I_k \in \mathcal{J}(\bigcup_{x \in \mathcal{O}_t} L_t^x) \right\}.$$

We decompose the integral defining $\Gamma_t(y)$ as

$$\begin{aligned}
\Gamma_t(y) &= \int_{\Omega_t^f \cap (-\infty, y]} (U_t(z) - U^{\text{free}}(z)) + \sum_{k \leq k_y} \int_{I_k} (U_t(z) - U_t^{\text{free}}(z)) \\
&= 0 + \sum_{k \leq k_y} \int_{I_k} (U_t(z) - U_t^{\text{free}}(z)) \\
&= \sum_{k \leq k_y} \left[\int_{I_k} \left(\frac{1}{|I_k|} \int_{I_k} U_t^{\text{free}}(z) \right) - \int_{I_k} V_0(z) \right] \\
&= 0
\end{aligned}$$

and thus

$$\Gamma_t = 0 \quad \text{on} \quad \mathbb{R} \setminus \bigcup_{x \in \mathcal{O}_t} L_t^x. \quad (43)$$

This in particular yields

$$\langle \partial_y \Gamma_t, S_t \rangle = 0. \quad (44)$$

Using (37), we obtain

$$\begin{aligned} \langle -\partial_y \Gamma_t, Z - S_t \rangle &= \langle U_t^{\text{free}} - U_t, Z - S_t \rangle \leq 0 \\ &\text{for all } Z \text{ in } K \text{ with } \langle X_t^{\text{free}} - X_t, Z - S_t \rangle = 0. \end{aligned}$$

Suppose that in addition $Z \in \mathcal{C}^1(\mathbb{R})$. Then

$$0 \geq \langle -\partial_y \Gamma_t, Z - S_t \rangle = \langle -\partial_y \Gamma_t, Z \rangle = \int_{\mathbb{R}} \Gamma_t(y) \partial_y Z(y) \, d\rho_0(y)$$

and we conclude that $\Gamma_t \leq 0$ a.e. \square

As in Proposition 2, we can define for a.e. $t \in (0, T)$ an Eulerian adhesion potential γ_t such that

$$\gamma_t(x) = \Gamma_t(y) \quad \text{with} \quad X_t(y) = x. \quad (45)$$

Exclusion relation. If ρ_t is a Borel family of probability measures satisfying the continuity equation in the distributional sense for a Borel velocity field u_t such that

$$\int_0^T \int_{\mathbb{R}} |u_t| \, d\rho_t(x) \, dt < +\infty,$$

then there exists a narrowly continuous curve $t \in [0, T] \mapsto \tilde{\rho}_t \in \mathcal{P}(\mathbb{R})$ such that

$$\rho_t = \bar{\rho}_t \quad \text{for a.e. } t \in (0, T);$$

see [1] or [22], for instance. To give a sense to the product $\gamma_t \rho_t$, we need to study the regularity in time of $t \mapsto \gamma_t$. Recall that the transport X_t satisfies a Lipschitz property which has allowed us to define the velocity U_t . We have then

$$X \quad \text{belongs to} \quad W^{1,\infty}(0, T; \mathcal{L}^2(\mathbb{R}, \rho_0)) \quad (46)$$

$$U \quad \text{belongs to} \quad L^\infty(0, T; \mathcal{L}^2(\mathbb{R}, \rho_0)), \quad (47)$$

which means that $\partial_y \Gamma$ is also in $L^\infty(0, T; \mathcal{L}^2(\mathbb{R}, \rho_0))$ and thus

$$\Gamma \quad \text{belongs to} \quad L^\infty(0, T; W^{1,2}(\mathbb{R}, \rho_0)).$$

We can then write

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \gamma_t(x) \, d\rho_t(x) \, dt &= \int_0^T \int_{\mathbb{R}} \Gamma_t(y) \, d\rho_0(y) \, dt \\ &= \int_0^T \int_{\Omega_{S_t}} \Gamma_t(y) \, d\rho_0(y) \, dt \\ &= \int_0^T \int_{\Omega_{S_t}} \Gamma_t(y) \frac{\partial_y X_t(y)}{\partial_y \tilde{X}(y)} \, d\rho_0(y) \, dt \\ &= \int_0^T \int_{\mathbb{R}} \Gamma_t(y) \frac{\partial_y X_t(y)}{\partial_y \tilde{X}(y)} \, d\rho_0(y) \, dt \\ &= \int_0^T \int_{\mathbb{R}} \Gamma_t(y) \partial_y X_t(y) \, dy \, dt \end{aligned}$$

since on Ω_{S_t} , S_t is constant and thus $\partial_y X_t(y) = \partial_y \tilde{X}(y)$. Finally this gives us the exclusion constraint by a change of variable

$$\int_0^T \int_{\mathbb{R}} \gamma_t(x) d\rho_t(x) dt = \int_0^T \int_{\mathbb{R}} \gamma_t(x) dx dt.$$

Since in addition we have $(1 - \rho_t)\gamma_t \leq 0$, we deduce that the exclusion relation holds a.e.:

$$(1 - \rho_t) \gamma_t = 0 \quad \text{a.e. on } (0, T) \times \mathbb{R}. \quad (48)$$

More generally we have

$$\int_0^T \int_{\mathbb{R}} \gamma_t(x) \phi(t, x) d\rho_t(x) dt = \int_0^T \int_{\mathbb{R}} \gamma_t(x) \phi(t, x) dx dt \quad \text{for all } \phi \in L^1(0, T; \mathcal{L}^2(\mathbb{R}, \rho_t)). \quad (49)$$

3.2 Recovering of the momentum equation

Similarly to the mass equation, we want to recover the momentum equation (5b) on the Eulerian variables by passing to the Lagrangian coordinates. For all $\varphi \in \mathcal{C}_c^\infty([0, T])$ we have

$$\begin{aligned} & - \int_{\mathbb{R}} \varphi(0, x) (u_0(x) - \partial_x \gamma_0(x)) d\rho_0(x) \\ &= \int_{\mathbb{R}} \varphi(0, X_0(y)) (U_0(y) - \partial_y \Gamma_0(y)) d\rho_0(y) \\ &= \int_0^T \frac{d}{dt} \int_{\mathbb{R}} \varphi(t, X_t(y)) (U_0(y) - \partial_y \Gamma_0(y)) d\rho_0(y) dt \\ &= \int_0^T \int_{\mathbb{R}} \frac{d}{dt} \varphi(t, X_t(y)) (U_0(y) - \partial_y \Gamma_0(y)) d\rho_0(y) dt. \end{aligned}$$

Replacing now $U_0(y) - \partial_y \Gamma_0(y)$ by $U_t - \partial_y \Gamma_t - \int_0^t f(s, X_s) ds$ we get

$$\begin{aligned} & - \int_{\mathbb{R}} \varphi(0, x) (u_0(x) - \partial_x \gamma_0(x)) d\rho_0(x) \\ &= \int_0^T \int_{\mathbb{R}} \frac{d}{dt} \varphi(t, X_t(y)) (U_t(y) - \partial_y \Gamma_t(y)) d\rho_0(y) dt \\ &\quad - \int_0^T \int_{\mathbb{R}} \frac{d}{dt} \varphi(t, X_t(y)) \left(\int_0^t f(s, X_s) ds \right) d\rho_0(y) dt \\ &= \int_0^T \int_{\mathbb{R}} \left(\partial_t \varphi(t, X_t(y)) + U_t(y) \partial_x \varphi(t, X_t(y)) \right) (U_t(y) - \partial_y \Gamma_t(y)) d\rho_0(y) dt \\ &\quad + \int_0^T \int_{\mathbb{R}} \varphi(t, X_t(y)) f(t, X_t(y)) d\rho_0(y) dt \end{aligned}$$

A final change of variable gives the weak formulation of the momentum equation (5b)

$$\begin{aligned} & - \int_{\mathbb{R}} \varphi(0, x) (u_0(x) - \partial_x \gamma_0(x)) d\rho_0(x) \\ &= \int_0^T \int_{\mathbb{R}} \left(\partial_t \varphi(t, x) + u_t(x) \partial_x \varphi(t, x) \right) v_t d\rho_t(x) dt + \int_0^T \int_{\mathbb{R}} \varphi(t, x) f(t, x) d\rho_t(x) dt \quad (50) \end{aligned}$$

which concludes our proof. \square

Remark: We have uniqueness among the class of weak solutions to (5) that can be written under the form

$$\rho_t = X_t \# \rho_0, \quad X_t = P_{\tilde{K}}(X_t^{\text{free}}).$$

Indeed, by the contraction property of the metric projection, for X_t^1, X_t^2 we have

$$\begin{aligned} \|X_t^1 - X_t^2\|_{\mathcal{L}_{\rho_0}^2} &= \left\| P_{\tilde{K}} \left(\text{id} + \int_0^t (U^{\text{free}})_\tau^1 d\tau \right) - P_{\tilde{K}} \left(\text{id} + \int_0^t (U^{\text{free}})_\tau^2 d\tau \right) \right\|_{\mathcal{L}_{\rho_0}^2} \\ &\leq \left\| \int_0^t (U^{\text{free}})_\tau^1 d\tau - \int_0^t (U^{\text{free}})_\tau^2 d\tau \right\|_{\mathcal{L}_{\rho_0}^2} \\ &\leq t \|V_0^1 - V_0^2\|_{\mathcal{L}^2} + k \int_0^t \int_0^\tau \|X_s^1 - X_s^2\|_{\mathcal{L}_{\rho_0}^2} ds d\tau \end{aligned}$$

where k is the Lipschitz constant associated to f . An analogue of the Gronwall inequality (see for instance [2] Theorem 11.4) shows that

$$\|X_t^1 - X_t^2\|_{\mathcal{L}_{\rho_0}^2} \leq t \|V_0^1 - V_0^2\|_{\mathcal{L}_{\rho_0}^2} + k \int_0^t \frac{s^2}{2} \exp(k(t-s)) \|V_0^1 - V_0^2\|_{\mathcal{L}_{\rho_0}^2} ds, \quad (51)$$

which proves that $X_t^1 = X_t^2$ if $V_0^1 = V_0^2$, and thus the uniqueness of the transport X_t . The velocity U_t is then uniquely defined since it is the orthogonal projection of

$$U_t^{\text{free}} = V_0 + \int_0^t f(s, X_s) ds$$

onto \mathcal{H}_{S_t} . Finally, the potential Γ_t is also unique by definition (42).

Remark: We can characterize a little bit better the convergence of our weak solution (ρ_t, u_t, γ_t) toward the initial data as $t \rightarrow 0$. For that purpose, let us define the L^2 -Wasserstein distance

$$W_2^2(\rho^1, \rho^2) := \int_{\mathbb{R}} |X^1(y) - X^2(y)|^2 d\rho_0(y) \quad \text{if } \rho^i = X_{\#}^i \rho_0$$

and, as in [17], the semi-distance

$$U_2^2((\rho^1, \rho^1 v^1), (\rho^2, \rho^2 v^2)) := \int_{\mathbb{R}} |v^1(X^1(y)) - v^2(X^2(y))|^2 d\rho_0(y).$$

Then $D_2((\rho^1, \rho^1 v^1), (\rho^2, \rho^2 v^2)) := W_2^2(\rho^1, \rho^2) + U_2^2((\rho^1, \rho^1 v^1), (\rho^2, \rho^2 v^2))$ is a distance on the space $\mathcal{V}_2(\mathbb{R}) := \left\{ (\rho, \rho v) \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) : v \in \mathcal{L}^2(\mathbb{R}, \rho) \right\}$. The space $(\mathcal{V}_2(\mathbb{R}), D_2)$ is metric but not complete, and its topology is stronger than the one induced by the weak convergence of measures (see [17] Proposition 2.1 and [1] Definition 5.4.3). One can prove that $(\rho_t, \rho_t v_t =$

$\rho_t(u_t - \partial_x \gamma_t)$ converges to $(\rho_0, \rho_0 v_0)$ in $\mathcal{V}_2(\mathbb{R})$ as $t \rightarrow 0$.
Indeed, the density ρ_t converges to ρ_0 for the Wasserstein distance

$$\begin{aligned} W_2^2(\rho_t, \rho_0) &= \int_{\mathbb{R}} |X_t(y) - X_0(y)|^2 d\rho_0(y) \\ &= \|X_t - \text{id}\|_{\mathcal{L}_{\rho_0}^2}^2 \\ &\leq t \|U_0 - \partial_y \Gamma_0\|_{\mathcal{L}_{\rho_0}^2}^2 + \left\| \int_0^t \int_0^\tau f(s, X_s) ds d\tau \right\|_{\mathcal{L}_{\rho_0}^2}^2 \longrightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned}$$

On the other hand, we get from the weak formulation of the momentum equation (50) the weak convergence in $\mathcal{M}(\mathbb{R})$ of $\rho_t v_t$ towards $\rho_0 v_0$. Finally

$$\begin{aligned} \int_{\mathbb{R}} |v_t(x)|^2 d\rho_t(x) &= \int_{\mathbb{R}} |U_t(y) - \partial_y \Gamma_t|^2 d\rho_0(y) \\ &\leq \|U_0 - \partial_y \Gamma_0\|_{\mathcal{L}_{\rho_0}^2}^2 + \int_0^t \|f_s\|_{L^\infty} ds \end{aligned}$$

and

$$\int_{\mathbb{R}} |v_t(x)|^2 d\rho_t(x) - \int_{\mathbb{R}} |v_0(x)|^2 d\rho_0(x) \leq t \|f\|_{L^\infty} \xrightarrow[t \rightarrow 0]{} 0$$

since $f \in L^\infty(0, T; \text{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. Since moreover

$$\int_{\mathbb{R}} |v_0(x)|^2 d\rho_0(x) \leq \liminf_{t \rightarrow 0^+} \int_{\mathbb{R}} |v_t(x)|^2 d\rho_t(x)$$

we get

$$\int_{\mathbb{R}} |v_t(x)|^2 d\rho_t(x) \rightarrow \int_{\mathbb{R}} |v_0(x)|^2 d\rho_0(x).$$

The combination of the convergence of ρ_t and those of $\rho_t v_t$ and $\|v_t\|_{\mathcal{L}_{\rho_t}^2}$ leads to the convergence of $(\rho_t, \rho_t v_t)$ to $(\rho_0, \rho_0 v_0)$ in $\mathcal{V}_2(\mathbb{R})$ (see [1] Definition 5.4.3).

4 Numerical simulation

To illustrate these memory effects in one-dimensional granular flows, we consider the initial data represented on Figure 1 formed by two congested blocks $\mathbf{1}_{[-1,0]}$ and $\mathbf{1}_{[0,1]}$ in contact at time $t = 0$. Initially, both velocity and adhesion potential are equal to zero.

We apply an external force f , such that the system first compresses and then decompresses in a second phase:

$$f(t, x) = \begin{cases} \alpha_1 & \text{if } x < 0 \\ -\alpha_1 & \text{if } x \geq 0 \end{cases} \quad \text{for } t \leq t^* \quad (52)$$

$$f(t, x) = \begin{cases} -\alpha_2 & \text{if } x < 0 \\ \alpha_2 & \text{if } x \geq 0 \end{cases} \quad \text{for } t > t^* \quad (53)$$

with α_1, α_2 two positive constants. In the simulation, $\alpha_1 = 1.10^{-3}$, $\alpha_2 = 1.10^{-2}$ and $t^* = 0.8$.

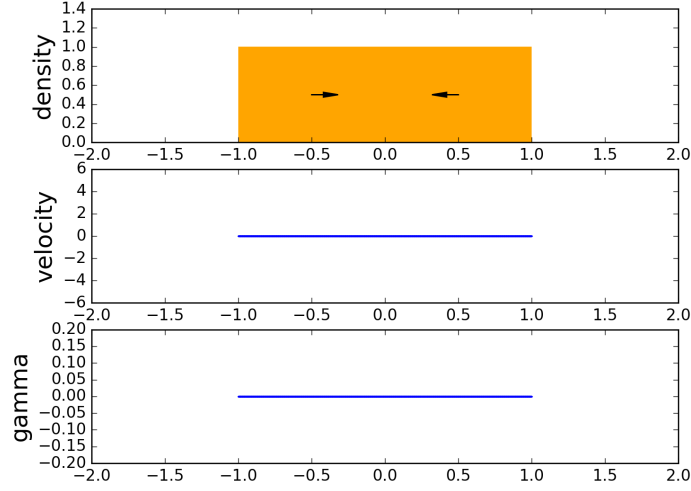


Figure 1: Initial data. The arrows represent the external force f at time 0.

Our numerical code follows the Lagrangian approach developed in the previous sections. To determine the transport X_{t_h} at time t_h , we are led to minimize the function

$$\phi_{t_h}(X) = \left\| X_0 + \int_0^{t_h} U_s^{\text{free}} ds - X \right\|_{L^2}^2$$

under the constraint $X \in \tilde{K}$. This step is performed by use of the Python software CVXOPT for convex optimization; see <http://cvxopt.org>. We have discretized the integral in time of the external force f by the left-hand rectangle method

$$\int_{t_h}^{t_h+\Delta t} f(s, X_s) ds \approx \Delta t f(t_h, X_{t_h}^\Delta)$$

where $X_{t_h}^\Delta \in \mathbb{R}^N$ denotes the numerical solution at the discrete time $t_h := h\Delta t$ with $\Delta t > 0$. Moreover we have expressed the constraint $X \in \tilde{K}$ through the linear constraint (recall that here we have chosen a congested initial density so that $\tilde{X} = \text{id}$)

$$GX_{t_h}^\Delta \leq X_0^\Delta$$

where the matrix $G \in \mathbb{R}^N \times \mathbb{R}^N$ is given by

$$G = \begin{pmatrix} 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix}$$

We observe on Figure 2 that in the congestion process while $t < t^* = 0.8$, the maximal density constraint $\rho_t \leq 1$ is satisfied, the velocity u_t is equal to 0 and the adhesion potential γ_t becomes negative for positive times. At time $t = t^*$ we reverse the external force f , and the

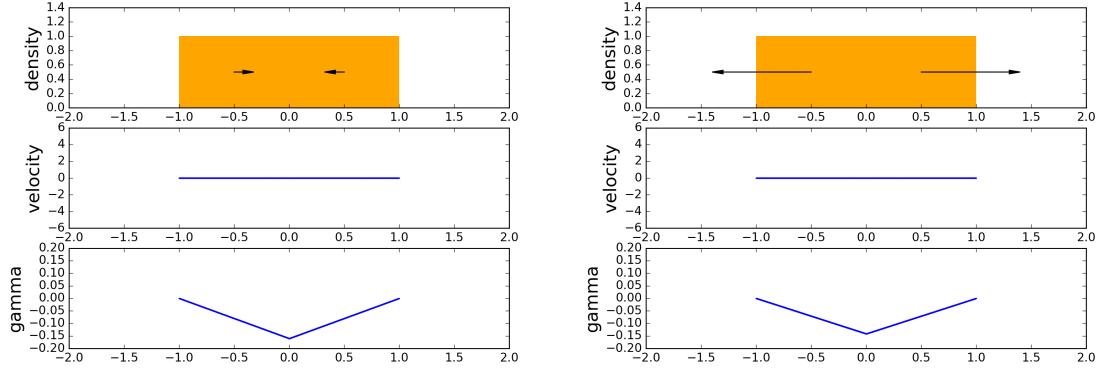


Figure 2: Granular system at times $t = 0.8$ (on the left) and $t = 1.0$ (on the right).

adhesion potential begins to increase (right picture on Figure 2), but the two blocks remain stuck since γ_t is still activated. This expresses the memory effects in the system.

At time $t \approx 1.16$ (left picture on Figure 3), the adhesion potential γ_t is 0. For larger times (right picture on Figure 3), the velocity u_t is non zero and we have a separation of the system into two blocks with opposite velocities.

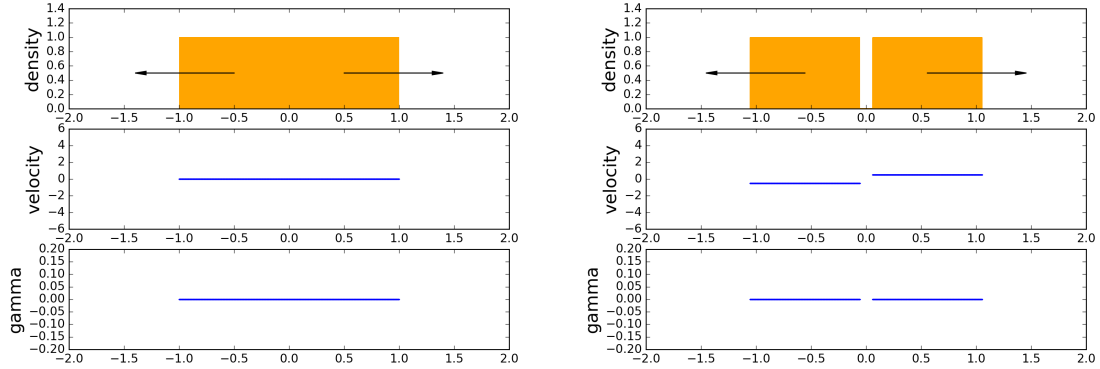


Figure 3: Granular system at times $t = 1.16$ (on the left) and $t = 1.4$ (on the right).

5 Extension to heterogeneous maximal constraint

We consider the case where the maximal density constraint ρ^* , previously supposed to be constant, is now transported by the velocity u

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (54a) \end{cases}$$

$$\begin{cases} \partial_t(\rho v) + \partial_x(\rho uv) = \rho f & (54b) \end{cases}$$

$$\begin{cases} v = u - \partial_x \gamma & (54c) \end{cases}$$

$$\begin{cases} 0 \leq \rho \leq \rho^*(t, x) & (54d) \end{cases}$$

$$\begin{cases} \partial_t \rho^* + u \partial_x \rho^* = 0 & (54e) \end{cases}$$

$$\begin{cases} (\rho^* - \rho) \gamma = 0, \quad \gamma \leq 0, & (54f) \end{cases}$$

as studied by Degond et al. [10] in the Navier-Stokes framework. Initially we prescribe, in addition to ρ_0, u_0, γ_0 , the initial constraint ρ_0^* with $\rho_0 \leq \rho^*$ a.e. in \mathbb{R} . Combining (54a) with (54e), we observe that the ratio $r = \frac{\rho}{\rho^*}$ is conserved by the flow

$$\partial_t r + \partial_x(ru) = 0. \quad (55)$$

Considering now the variable r instead of ρ , we are led to define the transport Y_t such that

$$r_t = (Y_t)_\# r_0$$

and to reformulate the system (54) as

$$\begin{cases} \partial_t r + \partial_x(ru) = 0 & (56a) \end{cases}$$

$$\begin{cases} \partial_t(rv) + \partial_x(ruv) = rf & (56b) \end{cases}$$

$$\begin{cases} v = u - \partial_x \gamma, \quad r = \frac{\rho}{\rho^*} & (56c) \end{cases}$$

$$\begin{cases} \partial_t \rho^* + u \partial_x \rho^* = 0 & (56d) \end{cases}$$

$$\begin{cases} 0 \leq r \leq 1 & (56e) \end{cases}$$

$$\begin{cases} (1 - r) \gamma = 0, \quad \gamma \leq 0 & (56f) \end{cases}$$

This transport Y_t has thus to satisfy the constraint $Y_t \in \tilde{K}$ with

$$\tilde{K} := K + \tilde{Y} \quad \text{with} \quad \tilde{Y} \quad \text{such that} \quad \tilde{\rho} = \tilde{Y}_\# r_0. \quad (57)$$

We can then construct in the same way as before a global weak solution to the heterogeneous system (56).

Theorem 2 *Let $T > 0$, $f \in L^\infty(0, T; \text{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, ρ_0 and ρ_0^* such that $\rho_0^* > 0$ and $r_0 = \frac{\rho_0}{\rho_0^*} \in \mathcal{P}_2(\mathbb{R})$, $r_0 \ll \mathcal{L}^1$ and $0 \leq \rho_0(x) \leq \rho_0^*(x)$ a.e. on \mathbb{R} . Moreover we require $(u_0, \gamma_0) \in \mathcal{L}^2(\mathbb{R}, r_0) \times W^{1,2}(\mathbb{R})$ with $(\rho_0^* - \rho_0)\gamma_0 = 0$ a.e. Let*

$$Y_0 := \text{id}, \quad U_0 := u_0 \circ Y_0, \quad \Gamma_0 := \gamma_0 \circ Y_0$$

and define for all $t \in [0, T]$ the coupled variables U_t^{free} and Y_t by

$$Y_t = P_{\tilde{K}} \left(Y_0 + \int_0^t U_s^{\text{free}} ds \right) \quad (58)$$

$$U_t = U_0 - \partial_y \Gamma_0 + \int_0^t f(s, Y_s) ds. \quad (59)$$

Then $t \mapsto Y_t$ is differentiable for a.e. $t \in (0, T)$ and we can define

$$U_t(y) := \dot{Y}_t(y), \quad \Gamma_t(y) := \int_{-\infty}^y \left(U_t(z) - U_t^{free}(z) \right) dr_0(z)$$

There exist $(u_t, \gamma_t) \in \mathcal{L}^2(\mathbb{R}, r_t) \times W^{1,2}(\mathbb{R})$ where $r_t := (Y_t)_\# r_0$ such that

$$U_t = u_t \circ Y_t, \quad \Gamma_t = \gamma_t \circ Y_t.$$

We set finally

$$\rho_t^*(x) = \rho_0^*(Y_t^{-1}(x)).$$

The triplet (r, u, γ, ρ^*) is a global weak solution of system (56).

References

- [1] AMBROSIO, L., GIGLI, N., AND SAVARÉ, G. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.
- [2] BAINOV, D. D., AND SIMEONOV, P. S. *Integral inequalities and applications*, vol. 57. Springer Science & Business Media, 2013.
- [3] BERTHELIN, F. Existence and weak stability for a pressureless model with unilateral constraint. *Mathematical Models and Methods in Applied Sciences* 12, 02 (2002), 249–272.
- [4] BOUCHUT, F. On zero pressure gas dynamics, advances in kinetic theory and computing, 171-190. *Ser. Adv. Math. Appl. Sci* 22 (1994).
- [5] BOUCHUT, F., BRENIER, Y., CORTES, J., AND RIPOLL, J.-F. A hierarchy of models for two-phase flows. *Journal of NonLinear Science* 10, 6 (2000), 639–660.
- [6] BOUDIN, L. A solution with bounded expansion rate to the model of viscous pressureless gases. *SIAM Journal on Mathematical Analysis* 32, 1 (2000), 172–193.
- [7] BRENIER, Y., GANGBO, W., SAVARÉ, G., AND WESTDICKENBERG, M. Sticky particle dynamics with interactions. *Journal de Mathématiques Pures et Appliquées* 99, 5 (2013), 577–617.
- [8] BRENIER, Y., AND GRENIER, E. Sticky particles and scalar conservation laws. *SIAM journal on numerical analysis* 35, 6 (1998), 2317–2328.
- [9] CAVALLETTI, F., SEDJRO, M., AND WESTDICKENBERG, M. A simple proof of global existence for the 1d pressureless gas dynamics equations. *SIAM Journal on Mathematical Analysis* 47, 1 (2015), 66–79.

- [10] DEGOND, P., MINAKOWSKI, P., AND ZATORSKA, E. Transport of congestion in the two-phase compressible/incompressible flow. *arXiv preprint arXiv:1612.08411* (2016).
- [11] JABIN, P.-E., AND REY, T. Hydrodynamic limit of granular gases to pressureless euler in dimension 1. *arXiv preprint arXiv:1602.09103* (2016).
- [12] LANNES, D. On the dynamics of floating structures. *arXiv preprint arXiv:1609.06136* (2016).
- [13] LEFEBVRE, A. *Modélisation numérique d'écoulements fluide-particules: prise en compte des forces de lubrification*. PhD thesis, Université de Paris-Sud. Faculté des Sciences d'Orsay (Essonne), 2007.
- [14] LEFEBVRE-LEPOT, A., AND MAURY, B. Micro-macro modelling of an array of spheres interacting through lubrication forces. *Advances in Mathematical Sciences and Applications* 21, 2 (2011), 535.
- [15] MAURY, B. A gluey particle model. In *ESAIM: Proceedings* (2007), vol. 18, EDP Sciences, pp. 133–142.
- [16] MAURY, B., AND PREUX, A. Pressureless Euler equations with maximal density constraint : a time-splitting scheme. <https://hal.archives-ouvertes.fr/hal-01224008> (2015).
- [17] NATILE, L., AND SAVARÉ, G. A wasserstein approach to the one-dimensional sticky particle system. *SIAM Journal on Mathematical Analysis* 41, 4 (2009), 1340–1365.
- [18] PERRIN, C. Modelling of phase transitions in one-dimensional granular flows. *To appear in ESAIM Procs* (2016).
- [19] PERRIN, C. Pressure-dependent viscosity model for granular media obtained from compressible navier–stokes equations. *Applied Mathematics Research eXpress* 2016, 2 (2016), 289–333.
- [20] PERRIN, C., AND ZATORSKA, E. Free/congested two-phase model from weak solutions to multi-dimensional compressible navier-stokes equations. *Communications in Partial Differential Equations* 40, 8 (2015), 1558–1589.
- [21] PREUX, A. *Transport optimal et équations des gaz sans pression avec contrainte de densité maximale*. Theses, Université Paris-Sud, Nov. 2016.
- [22] SANTAMBROGIO, F. *Optimal transport for applied mathematicians*, vol. 87. Springer, 2015.
- [23] SOBOLEVSKII, A. N. The small viscosity method for a one-dimensional system of equations of gas dynamic type without pressure. *Dokl. Akad. Nauk* 356, 3 (1997), 310–312.
- [24] WOLANSKY, G. Dynamics of a system of sticking particles of finite size on the line. *Nonlinearity* 20, 9 (2007), 2175.
- [25] ZEL'DOVICH, Y. B. Gravitational instability: An approximate theory for large density perturbations. *Astronomy and astrophysics* 5 (1970), 84–89.